

On the Order of Polynilpotent Multipliers of Some Nilpotent Products of Cyclic p -Groups

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Abstract. In this article we show that if \mathcal{V} is the variety of polynilpotent groups of class row (c_1, c_2, \dots, c_s) , $\mathcal{N}_{c_1, c_2, \dots, c_s}$, and $G \cong \mathbf{Z}_{p^{\alpha_1}} \ast^n \mathbf{Z}_{p^{\alpha_2}} \ast^n \dots \ast^n \mathbf{Z}_{p^{\alpha_t}}$ is the n th nilpotent product of some cyclic p -groups, where $c_1 \geq n$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $(q, p) = 1$ for all primes q less than or equal to n , then $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{d_m}$ if and only if $G \cong \mathbf{Z}_p \ast^n \mathbf{Z}_p \ast^n \dots \ast^n \mathbf{Z}_p$ (m -copies), where $m = \sum_{i=1}^t \alpha_i$ and $d_m = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(m)))) \dots$. Also, we extend the result to the multiple nilpotent product $G \cong \mathbf{Z}_{p^{\alpha_1}} \ast^{n_1} \mathbf{Z}_{p^{\alpha_2}} \ast^{n_2} \dots \ast^{n_{t-1}} \mathbf{Z}_{p^{\alpha_t}}$, where $c_1 \geq n_1 \geq \dots \geq n_{t-1}$. Finally a similar result is given for the c -nilpotent multiplier of $G \cong \mathbf{Z}_{p^{\alpha_1}} \ast^n \mathbf{Z}_{p^{\alpha_2}} \ast^n \dots \ast^n \mathbf{Z}_{p^{\alpha_t}}$ with the different conditions $n \geq c$ and $(q, p) = 1$ for all primes q less than or equal to $n + c$.

Keywords: Polynilpotent multiplier; Nilpotent product; Cyclic group; Finite p -group; Elementary Abelian p -group.

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1 Introduction and Motivation

Let G be any group with a presentation $G \cong F/R$, where F is a free group. Then the Baer invariant of G with respect to the variety of groups \mathcal{V} , denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

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where V is the set of words of the variety \mathcal{V} , $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1, \dots, f_{i-1}, f_i r, f_{i+1}, \dots, f_n) v(f_1, \dots, f_i, \dots, f_n)^{-1} | \\ r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in \mathbf{N} \rangle.$$

One may check that $\mathcal{VM}(G)$ is abelian and independent of the choice of the free presentation of G . In particular, if \mathcal{V} is the variety of abelian groups, \mathcal{A} , then the Baer invariant of the group G will be $(R \cap F')/[R, F]$, which is isomorphic to the well-known notion the Schur multiplier of G , denoted by $M(G)$. If \mathcal{V} is the variety of polynilpotent groups of class row (c_1, \dots, c_s) , $\mathcal{N}_{c_1, c_2, \dots, c_s}$, then the Baer invariant of a group G with respect to this variety, which is called a polynilpotent multiplier of G , is as follows:

$$\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) = \frac{R \cap \gamma_{c_s+1} \circ \dots \circ \gamma_{c_1+1}(F)}{[R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}(F), \dots, {}_{c_s}\gamma_{c_{s-1}+1} \circ \dots \circ \gamma_{c_1+1}(F)]},$$

where $\gamma_{c_s+1} \circ \dots \circ \gamma_{c_1+1}(F) = \gamma_{c_s+1}(\gamma_{c_{s-1}+1}(\dots(\gamma_{c_1+1}(F))\dots))$ are the term of iterated lower central series of F . See Hekster [6] for the equality

$$[R\mathcal{N}_{c_1, c_2, \dots, c_s}^* F] = [R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}(F), \dots, {}_{c_s}\gamma_{c_{s-1}+1} \circ \dots \circ \gamma_{c_1+1}(F)].$$

In particular, if $s = 1$ and $c_1 = c$, then the Baer invariant of G with respect to the variety \mathcal{N}_c , which is called the c -nilpotent multiplier of G , is

$$\mathcal{N}_c M(G) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]}.$$

Historically, Green [4] showed that the order of the Schur multiplier of a finite p -group of order p^n is bounded by $p^{\frac{n(n-1)}{2}}$. Berkovich [2] showed that a finite p -group of order p^n is an elementary abelian p -group if and only if the order of $M(G)$ is $p^{n(n-1)/2}$. Moghaddam [15,16] presented a bound for the polynilpotent multiplier of a finite p -group. He showed that if \mathcal{V} is the variety of polynilpotent groups of a given class row and G is a finite d -generator group of order p^n , then

$$|\mathcal{VM}(\mathbf{Z}_p^{(d)})| \leq |\mathcal{VM}(G)| |V(G)| \leq |\mathcal{VM}(\mathbf{Z}_p^{(n)})|,$$

where $\mathbf{Z}_p^{(m)}$ denotes the direct sum of m copies of \mathbf{Z}_p . As a consequence, using the structure of $\mathcal{VM}(\mathbf{Z}_p^{(n)})$ in [12], we can show that the order of the nilpotent multiplier of a finite p -group of order p^n is bounded by $p^{\chi_{c+1}(n)}$, where $\chi_{c+1}(n)$ is the number of basic commutators of weight $c+1$ on n letters. The first author and Sanati [13] extended a result of Berkovich to the c -nilpotent multiplier of a finite p -group. They showed that for an abelian p -group G , $|\mathcal{N}_c M(G)| = p^{\chi_{c+1}(n)}$ if and only if G is an elementary abelian p -group. Putting an additional condition on the kernel of the left natural map of the generalized Stallings-Stammbach five-term exact sequence, they showed that an arbitrary finite p -group with the c -nilpotent multiplier of maximum order is an elementary abelian p -group.

Unfortunately, there is a mistake in the proof of Theorem 3.5 in [13] due to using the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i+1)$ which is not correct in general. In this paper, first, we give a correct proof for Theorem 3.5 in [13]. Second, we extend the result in different directions. In fact, we show that if \mathcal{V} is the variety of polynilpotent groups of class row (c_1, c_2, \dots, c_s) , $\mathcal{N}_{c_1, c_2, \dots, c_s}$, and $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ is the n th nilpotent product of some cyclic p -groups, where $c_1 \geq n$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $(q, p) = 1$ for all primes q less than or equal to n , then $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{d_m}$ if and only if $G \cong \mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_p$ (m -copies), where $m = \sum_{i=1}^t \alpha_i$ and $d_m = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(m)))\dots)$. Also, we extend the above result to the multiple nilpotent product of cyclic p -groups $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n_1}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n_2}{*} \dots \overset{n_{t-1}}{*} \mathbf{Z}_{p^{\alpha_t}}$, when $c_1 \geq n_1 \geq \dots \geq n_{t-1}$. As a consequence we show that the polynilpotent multiplier of a finite abelian p -group G has maximum order if and only if G is an elementary abelian p -group. Finally we give a similar result for the c -nilpotent multiplier of $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$, with the different conditions $n \geq c$ and $(q, p) = 1$ for all primes q less than or equal to $n + c$.

2 Notation and Preliminaries

Definition 2.1. Let $\{G_i\}_{i \in I}$ be a family of arbitrary groups. The n th nilpotent product of the family $\{G_i\}_{i \in I}$ is defined as follows:

$$\prod_{i \in I}^n G_i = \frac{\prod_{i \in I}^* G_i}{\gamma_{n+1}(\prod_{i \in I}^* G_i) \cap [G_i]_{i \in I}^*},$$

where $\prod_{i \in I}^* G_i$ is the free product of the family $\{G_i\}_{i \in I}$, and

$$[G_i]_{i \in I}^* = \langle [G_i, G_j] | i, j \in I, i \neq j \rangle \prod_{i \in I}^* G_i$$

is the cartesian subgroup of the free product $\prod_{i \in I}^* G_i$ which is the kernel of the natural homomorphism from $\prod_{i \in I}^* G_i$ to the direct product $\prod_{i \in I}^\times G_i$. For further properties of the above notation see Neumann [17]. If $\{G_i\}_{i \in I}$ is a family of cyclic groups, then $\gamma_{n+1}(\prod_{i \in I}^* G_i) \subseteq [G_i]^*$ and hence $\prod_{i \in I}^n G_i = \prod_{i \in I}^* G_i / \gamma_{n+1}(\prod_{i \in I}^* G_i)$.

Definition 2.2. A variety \mathcal{V} is said to be a Schur-Baer variety if for any group G for which the marginal factor group $G/V^*(G)$ is finite, then the verbal subgroup $V(G)$ is also finite and $|V(G)|$ divides a power of $|G/V^*(G)|$.

Schur proved that the variety of abelian groups, \mathcal{A} , is a Schur-Baer variety (see [10]). Also, Baer [1] proved that if u and v have Schur-Baer property, then the variety defined by the word $[u, v]$ has the above property.

The following theorem gives a very important property of Schur-Baer varieties.

Theorem 2.3. (Leedham-Green, McKay [11]). The following conditions on a variety \mathcal{V} are equivalent:

- (i) \mathcal{V} is a Schur-Baer variety.
- (ii) For every finite group G , its Baer invariant, $\mathcal{V}M(G)$, is of order dividing a power of $|G|$.

In the rest of this section we review some theorems required in the proofs of the main results of the article.

Theorem 2.4. (Jones [9]). Let G be a finite d -generator group of order p^n . Then

$$p^{\frac{1}{2}d(d-1)} \leq |G'| |M(G)| \leq p^{\frac{1}{2}n(n-1)}.$$

Theorem 2.5. (Berkovich [2]). Let G be a finite group of order p^n . Then $|M(G)| = p^{\frac{1}{2}n(n-1)}$ if and only if G is an elementary abelian p -group.

Theorem 2.6. (Moghaddam [15,16]). Let \mathcal{V} be the variety of polynilpotent groups of a given class row. Let G be a finite d -generator group of order p^n . Then

$$|\mathcal{V}M(\mathbf{Z}_p^{(d)})| \leq |\mathcal{V}M(G)| |V(G)| \leq |\mathcal{V}M(\mathbf{Z}_p^{(n)})|.$$

We recall that the number of basic commutators of weight c on n generators, denoted by $\chi_c(n)$, is determined by Witt formula (see [5]).

Theorem 2.7. (Moghaddam and Mashayekhy [14]). Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \dots \oplus \mathbf{Z}_{n_k}$ be a finite abelian groups, where n_{i+1} divides n_i for all $1 \leq i \leq k-1$. Then for all $c \geq 1$, the c -nilpotent multiplier of G is

$$\mathcal{N}_c M(G) = \mathbf{Z}_{n_2}^{(b_2)} \oplus \mathbf{Z}_{n_3}^{(b_3-b_2)} \oplus \dots \oplus \mathbf{Z}_{n_k}^{(b_k-b_{k-1})},$$

where $b_i = \chi_{c+1}(i)$.

The following result is an interesting consequence of Theorems 2.6 and 2.7.

Corollary 2.8. Let G be a finite d -generator p -group of order p^n , then

$$p^{\chi_{c+1}(d)} \leq |\mathcal{N}_c M(G)| |\gamma_{c+1}(G)| \leq p^{\chi_{c+1}(n)}.$$

The following theorem is a generalization of Theorem 2.5.

Theorem 2.9. (Mashayekhy, Sanati [13]). Let G be an abelian group of order p^n . Then $\mathcal{N}_c M(G) = p^{\chi_{c+1}(n)}$ if and only if G is an elementary abelian p -group.

Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a \mathcal{V} -central extension, where \mathcal{V} is any variety of groups, that is, the above sequence is exact and N is contained in the marginal subgroup of G , $V^*(G)$. Then the following five-term exact sequence exists (see Fröhlich [3]):

$$\mathcal{V}M(G) \xrightarrow{\theta} \mathcal{V}M(Q) \rightarrow N \rightarrow G/V(G) \rightarrow Q/V(Q) \rightarrow 1.$$

The nonabelian version of Theorem 2.9 is as follows.

Theorem 2.10. (Mashayekhy, Sanati [13]). Let G be a finite p -group of order p^n . If $\mathcal{N}_c M(G) = p^{\chi_{c+1}(n)}$, then

- (i) There is an epimorphism $\mathcal{N}_c M(G) \xrightarrow{\theta} \mathcal{N}_c M(G/G')$ which is obtained from the Fröhlich sequence.
- (ii) If $\ker(\theta) = 1$, then G is an elementary abelian p -group.

Theorem 2.11. (Mashayekhy, Parvizi [12]). Let

$$G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \dots \oplus \mathbf{Z}_{n_k}$$

be a finitely generated abelian group, where n_{i+1} divides n_i for all $1 \leq i \leq k-1$. Then

$$\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) \cong \mathbf{Z}^{(\beta_m)} \oplus \mathbf{Z}_{n_1}^{(\beta_{m+1}-\beta_m)} \oplus \dots \oplus \mathbf{Z}_{n_k}^{(\beta_{m+k}-\beta_{m+k-1})},$$

where $\beta_i = \chi_{c_s+1}(\chi_{c_{s-1}+1}(\dots(\chi_{c_1+1}(i))\dots))$ for all $m \leq i \leq m+k$.

Theorems 2.6 and 2.11 imply the following useful inequalities.

Corollary 2.12. With the notation of previous theorem let G be a finite d -generator p -group of order p^n . Then

$$p^{\beta_d} \leq |\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)|^{\gamma_{c_s+1}(\gamma_{c_{s-1}+1}(\dots(\gamma_{c_1+1}(G))\dots))} \leq p^{\beta_n}.$$

Theorem 2.13. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong \underbrace{\mathbf{Z}^n * \dots * \mathbf{Z}^n}_{m\text{-copies}}$

$\mathbf{Z}_{r_1}^n * \dots * \mathbf{Z}_{r_t}^n$, be the n th nilpotent product of some cyclic groups, where r_{i+1} divides r_i for all $1 \leq i \leq t-1$. If $c \geq n$ and $(p, r_1) = 1$ for all primes p less than or equal to n , then the c -nilpotent multiplier of G is isomorphic to

$$\mathbf{Z}^{(\sum_{i=1}^n \chi_{c+i}(m))} \oplus \mathbf{Z}_{r_1}^{(\sum_{i=1}^n (\chi_{c+i}(m+1) - \chi_{c+i}(m)))} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(\sum_{i=1}^n (\chi_{c+i}(m+t) - \chi_{c+i}(m+t-1)))}.$$

Theorem 2.14. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong \underbrace{\mathbf{Z}^n * \dots * \mathbf{Z}^n}_{m\text{-copies}}$

$\mathbf{Z}_{r_1}^n * \dots * \mathbf{Z}_{r_t}^n$ be the n th nilpotent product of some cyclic groups, where r_{i+1} divides r_i for all $1 \leq i \leq t-1$. If $(p, r_1) = 1$ for all primes p less than or equal to n , then the structure of the polynilpotent multiplier of G is

$$\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) = \mathbf{Z}^{(d_m)} \oplus \mathbf{Z}_{r_1}^{(d_{m+1}-d_m)} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(d_{m+t}-d_{m+t-1})},$$

where $d_i = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(i)))\dots)$, for all $c_1 \geq n$ and $c_2, \dots, c_s \geq 1$ and $m \leq i \leq m+t$.

Theorem 2.15. (Hokmabadi, Mashayekhy [7]). Let $G \cong \underbrace{\mathbf{Z}^n * \dots * \mathbf{Z}^n}_{m\text{-copies}} * \mathbf{Z}_{r_1}^n * \dots * \mathbf{Z}_{r_t}^n$

be the n th nilpotent product of some cyclic groups such that r_{i+1} divides r_i for all $1 \leq i \leq t-1$. If $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, then

- (i) if $n \geq c$, then $\mathcal{N}_c M(G) = \mathbf{Z}^{(g_0)} \oplus \mathbf{Z}_{r_1}^{(g_1-g_0)} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(g_t-g_{t-1})}$;
- (ii) if $c \geq n$, then $\mathcal{N}_c M(G) = \mathbf{Z}^{(f_0)} \oplus \mathbf{Z}_{r_1}^{(f_1-f_0)} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(f_t-f_{t-1})}$,

where $f_k = \sum_{i=1}^n \chi_{c+i}(m+k)$ and $g_k = \sum_{i=1}^c \chi_{n+i}(m+k)$ for $0 \leq k \leq t$.

Theorem 2.16. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong A_1 \overset{n_1}{*} A_2 \overset{n_2}{*} \dots \overset{n_k}{*} A_{k+1}$ such that $A_i \cong \mathbf{Z}$ for $1 \leq i \leq t$ and $A_j \cong \mathbf{Z}_{m_j}$ for $t+1 \leq j \leq k+1$. Let $c_1 \geq n_1 \geq n_2 \geq \dots \geq n_k$ and $m_{k+1} | m_k | \dots | m_{t+1}$ and $(p, m_{t+1}) = 1$ for all primes $p \leq n_1$. Then the structure of the polynilpotent multiplier of G is

$$\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) = \mathbf{Z}^{(e_0)} \oplus \mathbf{Z}_{m_{t+1}}^{(e_t - e_0)} \oplus \dots \oplus \mathbf{Z}_{m_{k+1}}^{(e_k - e_{k-1})},$$

where $e_i = \chi_{c_s+1}(\dots(\chi_{c_2+1}(u + \sum_{j=t}^i h_j))\dots)$, for all $t \leq i \leq k$, $e_0 = \chi_{c_s+1}(\dots(\chi_{c_2+1}(u))\dots)$, $u = \sum_{j=1}^{n_{t-1}} \chi_{c_1+j}(t) + \sum_{i=1}^{t-2} \sum_{j=n_{i+1}+1}^{n_i} \chi_{c_1+j}(i+1)$ and $h_j = \sum_{\lambda=1}^{n_j} (\chi_{c_1+\lambda}(j+1) - \chi_{c_1+\lambda}(j))$.

3 Main Results

As we mentioned before, there is a mistake in the proof of Theorem 2.9. More precisely, in the proof of Theorem 3.5 in [13] it is assumed that G is a finite abelian d -generator p -group of order p^n and $|\mathcal{N}_c M(G)| = p^{\chi_{c+1}(n)}$. Then using the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i+1)$ it is proved that $n = d$ and therefore G is an elementary abelian p -group. Unfortunately, the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i+1)$ is not correct and so the proof is not valid.

In this section, first, we intend to present a new proof for Theorem 2.9 in order to remedy the above mentioned mistake. Second, using this new method, we extend the result to polynilpotent multipliers of nilpotent products of cyclic p -groups with some conditions.

Proof of Theorem 2.9.

Proof. Let G be an elementary abelian p -group of order p^n . Then by Theorem 2.7 we have $\mathcal{N}_c M(G) = \mathbf{Z}_p^{(b_2)} \oplus \mathbf{Z}_p^{(b_3-b_2)} \oplus \dots \oplus \mathbf{Z}_p^{(b_n-b_{n-1})}$, where $b_i = \chi_{c+1}(i)$, and hence $|\mathcal{N}_c M(G)| = p^{\chi_{c+1}(n)}$.

Conversely, suppose that $|\mathcal{N}_c M(G)| = p^{\chi_{c+1}(n)}$. Since G is an abelian p -group of order p^n , we can consider G as follows:

$$G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_d}},$$

where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d$ and $\alpha_1 + \alpha_2 + \dots + \alpha_d = n$. By Theorem 2.7 $\mathcal{N}_c M(G) = \mathbf{Z}_{p^{\alpha_2}}^{(b_2)} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(b_3-b_2)} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_d}}^{(b_d-b_{d-1})}$ and so $|\mathcal{N}_c M(G)| = p^{\alpha_2 b_2 + \alpha_3(b_3-b_2) + \dots + \alpha_d(b_d-b_{d-1})}$. On the other hand by hypothesis $|\mathcal{N}_c M(G)| = p^{b_n}$. Therefore $b_n = \alpha_2 b_2 + \alpha_3(b_3-b_2) + \dots + \alpha_d(b_d-b_{d-1})$. Also $b_n = (b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \dots + (b_3 - b_2) + b_2$. Thus

$$(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \dots + (b_3 - b_2) + b_2 = \alpha_2 b_2 + \alpha_3(b_3 - b_2) + \dots + \alpha_d(b_d - b_{d-1}) =$$

$$\underbrace{b_2 + \dots + b_2}_{\alpha_2\text{-copies}} + \underbrace{(b_3 - b_2) + \dots + (b_3 - b_2)}_{\alpha_3\text{-copies}} + \dots + \underbrace{(b_d - b_{d-1}) + \dots + (b_d - b_{d-1})}_{\alpha_d\text{-copies}}.$$

So we have the following equality:

$$(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \dots + (b_{d+1} - b_d) =$$

$$\underbrace{b_2 + \dots + b_2}_{\alpha_2-1\text{-copies}} + \underbrace{(b_3 - b_2) + \dots + (b_3 - b_2)}_{\alpha_3-1\text{-copies}} + \dots + \underbrace{(b_d - b_{d-1}) + \dots + (b_d - b_{d-1})}_{\alpha_d-1\text{-copies}} \quad (I).$$

One can easily see that for any $i \geq 1$, $(b_i - b_{i-1})$ is the number of basic commutators of weight $c + 1$ on i letters such that x_i does appear in it. So $(b_j - b_{j-1}) \geq (b_i - b_{i-1})$ whenever $j \geq i$. Now, assume $\alpha_1 \geq 2$. Then $n - 1 > n - \alpha_1$ and so the left-hand side of the above equality has more terms than the right-hand side. Also each term of the left-hand side of the above equality is greater than of any term of the right-hand side. These facts imply that the equality (I) does not hold which is a contradiction. Thus we must have $\alpha_1 = 1$ and hence $d = n$, $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$. Therefore the result holds. \square

The next theorem is a generalization of Theorem 2.9. Note that the nilpotent product of finitely many finite p -groups is also a finite p -group.

Theorem 3.1. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ be the n th nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $(q, p) = 1$ for all primes q less than or equal to n . Let $\mathcal{N}_{c_1, c_2, \dots, c_s}$ be a variety of polynilpotent groups such that $c_1 \geq n$. Then $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{d_m}$ if and only if $G \cong \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_p}_{m\text{-copies}}$, where $m = \sum_{i=1}^t \alpha_i$ and

$$d_m = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(m)))\dots).$$

Proof. Let $G = \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_p}_{m\text{-copies}}$ and $(q, p) = 1$ for all primes q less than or equal to n .

Then by Theorem 2.14,

$$\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) = \mathbf{Z}_p^{(d_2)} \oplus \dots \oplus \mathbf{Z}_p^{(d_m - d_{m-1})},$$

where $d_i = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(i)))\dots)$, for all $c_1 \geq n$. Hence

$$|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{d_m}.$$

Conversely, suppose that $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{d_m}$. By the hypothesis $G = \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $\alpha_1 + \alpha_2 + \dots + \alpha_t = m$. Now Theorem 2.14 implies that

$$\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) = \mathbf{Z}_{p^{\alpha_2}}^{(d_2)} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(d_3 - d_2)} \dots \oplus \mathbf{Z}_{p^{\alpha_t}}^{(d_t - d_{t-1})},$$

where $d_i = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^n \chi_{c_1+j}(i)))\dots)$. Thus

$$|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{\alpha_2 d_2 + \alpha_3 (d_3 - d_2) + \dots + \alpha_t (d_t - d_{t-1})}.$$

On the other hand by hypothesis $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{d_m}$. Therefore $d_m = \alpha_2 d_2 + \alpha_3 (d_3 - d_2) + \dots + \alpha_t (d_t - d_{t-1})$. Now applying a similar method to the proof of Theorem 2.9, it is enough to show that if $j \geq i$, then $(d_j - d_{j-1}) \geq (d_i - d_{i-1})$. In order to prove this fact consider the following sets:

$$A_1 = \{\alpha | \alpha \text{ is a basic commutator of weight } c_1 + 1, \dots, c_1 + n \text{ on } x_1, \dots, x_i \}$$

and inductively for all $2 \leq k \leq s$

$$A_k = \{\alpha | \alpha \text{ is a basic commutator of weight } c_k + 1 \text{ on } A_{k-1}\}.$$

Clearly $d_i = |A_s|$. It is easy to see that

$$d_i - d_{i-1} = |\{\alpha | \alpha \text{ is a basic commutator of weight } c_s + 1 \text{ on } A_{s-1} \text{ such that } x_i \text{ does appear in } \alpha \}|.$$

Hence the required inequality holds. \square

Using Theorem 2.16 and a similar proof to the above and noting that $j \geq i$ implies $e_j - e_{j-1} \geq e_i - e_{i-1}$, we can state the following theorem.

Theorem 3.2. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} *^{n_1} \mathbf{Z}_{p^{\alpha_2}} *^{n_2} \dots *^{n_{t-1}} \mathbf{Z}_{p^{\alpha_t}}$ be the n th nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $(q, p) = 1$ for all primes q less than or equal to n . Let $\mathcal{N}_{c_1, c_2, \dots, c_s}$ be a variety of polynilpotent groups such that $c_1 \geq n_1$. Then $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{e_{m-1}}$ if and only if $G \cong \underbrace{\mathbf{Z}_p *^{n_1} \mathbf{Z}_p *^{n_2} \dots *^{n_{m-1}} \mathbf{Z}_p}_{m\text{-copies}}$, where $m = \sum_{i=1}^t \alpha_i$, $e_{m-1} = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=0}^{m-1} h_j))\dots)$, and $h_j = \sum_{\lambda=1}^{n_j} (\chi_{c_1+\lambda}(j+1) - \chi_{c_1+\lambda}(j))$.

The following result is a consequence of Theorem 2.15 and the above mentioned method with different condition $n \geq c$.

Theorem 3.3. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} *^n \mathbf{Z}_{p^{\alpha_2}} *^n \dots *^n \mathbf{Z}_{p^{\alpha_t}}$ be the n th nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$, $n \geq c$ and $(q, p) = 1$ for all primes q less than or equal to $n + c$. Then $\mathcal{N}_c M(G) = p^{g_m}$ if and only if $G \cong \underbrace{\mathbf{Z}_p *^n \mathbf{Z}_p *^n \dots *^n \mathbf{Z}_p}_{m\text{-copies}}$, where

$$m = \sum_{i=1}^t \alpha_i \text{ and } g_m = \sum_{i=1}^c \chi_{n+i}(m).$$

With the assumption and notation of Theorem 3.1, let $n = 1$. Then the n th nilpotent product of $\mathbf{Z}_{p^{\alpha_i}}$ ($1 \leq i \leq t$) is the direct product of $\mathbf{Z}_{p^{\alpha_i}}$. So G is a finite abelian p -group of order p^m . Also d_i will be equal to β_i in Theorem 2.12. Therefore the following corollary is a consequence of Theorem 3.1.

Corollary 3.4. Let G be an abelian group of order p^m . Then $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{\beta_m}$ if and only if G is an elementary abelian p -group, where

$$\beta_m = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\chi_{c_1+1}(m)))\dots).$$

Note that according to Corollary 2.12 the polynilpotent multiplier of G in the above result has its maximum order. So the above corollary is a vast generalization of Theorem 2.5.

Finally in order to deal with a non-abelian case we present the following theorem. This theorem is a generalization of Theorem 2.10.

Theorem 3.5. With the previous notation let G be a finite p -group of order p^m . If $\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) = p^{\beta_m}$, then the following statements holds.

- (i) There is an epimorphism $\mathcal{N}_{c_1, c_2, \dots, c_s} M(G) \xrightarrow{\theta} \mathcal{N}_{c_1, c_2, \dots, c_s} M(G/G')$ which is obtained from the Fröhlich sequence.
- (ii) If $\ker(\theta) = 1$, then G is an elementary abelian p -group.

Proof. (i) Let \mathcal{V} be the variety of polynilpotent groups of class row (c_1, c_2, \dots, c_s) , $\mathcal{N}_{c_1, c_2, \dots, c_s}$. By Corollary 2.12 we have $|\mathcal{V}M(G)||V(G)| \leq p^{\beta_m}$. Also by the hypothesis $|\mathcal{N}_{c_1, c_2, \dots, c_s} M(G)| = p^{\beta_m}$. Therefore $|V(G)| = 1$. Now set $N = G'$ and consider the exact sequence $1 \rightarrow G' \rightarrow G \rightarrow G/G' \rightarrow 1$. Since $|V(G)| = 1$, the above sequence is an \mathcal{V} -central extension. Therefore by Fröhlich five-term exact sequence we have the following exact sequence:

$$\mathcal{V}M(G) \xrightarrow{\theta} \mathcal{V}M(G/G') \xrightarrow{\beta} G' \xrightarrow{\alpha} G \rightarrow G/G' \rightarrow 1.$$

Clearly α is a monomorphism and so $\text{Im}(\beta) = 1$. This means that θ is an epimorphism. (ii) Let $\ker(\theta) = 1$. Then $|\mathcal{V}M(G/G')| = |\mathcal{V}M(G)| = p^{\beta_m}$ (*). Since $|G| = p^m$ then $|G/G'| \leq p^m$. Hence $|G/G'| = p^m$, otherwise, if $|G/G'| = p^k$, where $k < m$, then $|\mathcal{V}M(G/G')| \leq |\mathcal{V}M(G/G')||V(G/G')| \leq p^{\beta_k} < p^{\beta_m}$, which is a contradiction to (*). Hence $|G/G'| = p^m$ which implies that G is an abelian p -group. Now by Corollary 3.4 the result holds. \square

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